

Random Aharonov-Bohm vortices and some exact families of integrals: Part II

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Abstract

At 6th order in perturbation theory, the random magnetic impurity problem at second order in impurity density narrows down to the evaluation of a single Feynman diagram with maximal impurity line crossing. This diagram can be rewritten as a sum of ordinary integrals and nested double integrals of products of the modified Bessel functions K_ν and I_ν , with $\nu = 0, 1$. That sum, in turn, is shown to be a linear combination with rational coefficients of $(2^5 - 1)\zeta(5)$, $\int_0^\infty u K_0(u)^6 du$ and $\int_0^\infty u^3 K_0(u)^6 du$. Unlike what happens at lower orders, these two integrals are not linear combinations with rational coefficients of Euler sums, even though they appear in combination with $\zeta(5)$. On the other hand, any integral $\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du$ with weight $p + q = 6$ and an even n is shown to be a linear combination with rational coefficients of the above two integrals and 1, a result that can be easily generalized to any weight $p + q = k$. A matrix recurrence relation in n is built for such integrals. The initial conditions are such that the asymptotic behavior is determined by the smallest eigenvalue of the transition matrix.

1 Introduction

In Ref. [1], the quantum problem of a charged particle in a plane, coupled to a random Poissonian distribution of infinitely thin impenetrable Aharonov-Bohm flux tubes (magnetic vortices) perpendicular to the plane was considered. The relevant parameters are $\alpha = \phi/\phi_0$, where ϕ is the flux of a tube and ϕ_0 the flux quantum, and the mean impurity density $\rho = N/V$, where N is the number of impurities and V the area (in the thermodynamic limit, $N, V \rightarrow \infty$ with N/V fixed). Periodicity $\alpha \rightarrow \alpha + 1$ and symmetry with respect to $\alpha = 1/2$ imply that the N -impurity partition function Z_N is invariant under $\alpha \rightarrow 1 - \alpha$, and depends only on $\alpha(1 - \alpha)$.

One was interested in the average partition function

$$\langle Z \rangle = e^{-\rho V} \sum_N \frac{(\rho V)^N}{N!} \langle Z_N \rangle \quad (1)$$

i.e.,

$$\begin{aligned} \frac{\langle Z \rangle}{Z_0} &= 1 + \rho V \left[\frac{\langle Z_1 \rangle}{Z_0} - 1 \right] + \frac{(\rho V)^2}{2!} \left[\frac{\langle Z_2 \rangle}{Z_0} - 2 \frac{\langle Z_1 \rangle}{Z_0} + 1 \right] \\ &+ \frac{(\rho V)^3}{3!} \left[\frac{\langle Z_3 \rangle}{Z_0} - 3 \frac{\langle Z_2 \rangle}{Z_0} + 3 \frac{\langle Z_1 \rangle}{Z_0} - 1 \right] + \dots \end{aligned} \quad (2)$$

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where Z_0 is the free partition function.

With account for the α dependence of Z_N , Eq. (2) becomes a sum of terms proportional to $\rho^n \alpha^m$ with $m \geq n$. For small α , the leading terms are $(\rho\alpha)^n$. They yield the partition function of the charge in the mean magnetic field $\rho\alpha\phi_0$, which replaces the local magnetic field $\phi \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i)$, \vec{r}_i being the location of the i -th impurity. Terms with $m > n$ are perturbative corrections to the mean-field expansion, which originate from disorder effects. For the 1-impurity case [2], which is exactly solvable, one obtains $\langle Z_1 \rangle - Z_0 = \alpha(\alpha - 1)/2$. For the 2-impurity case, nontrivial Feynman diagrams with maximal impurity line crossing appear at order $\rho^2 \alpha^4$, i.e., an electron interacting 4 times with 2 impurities; at order $\rho^2 \alpha^6$, i.e., an electron interacting 6 times with 2 impurities, etc. Knowing the $\rho^2 \alpha^4$ Feynman diagram is sufficient to get a rather precise analytical estimate of the critical disorder coupling constant $\alpha_c \simeq 0.35$, above which oscillations in the density of states, corresponding to the Landau levels in the mean magnetic field picture, disappear. That is a clear indication that the system becomes fully disordered. Note, on the other hand, that at weak disorder, when $\alpha \leq \alpha_c$, the broadening of the Landau levels due to disorder fits nicely into the Integer Quantum Hall Effect paradigm.

In Ref. [1], the $\rho^2 \alpha^4$ diagram was reduced to a multiple temperature integral

$$I_{\rho^2 \alpha^4} = \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \int_0^{\beta_2} d\beta_3 \int_0^{\beta_3} d\beta_4 \left(\frac{2}{\beta} - \frac{(a+c)(b+d)}{abc + bcd + cda + dab} \right) \quad (3)$$

where $a = \beta_1 - \beta_2$, $b = \beta_2 - \beta_3$, $c = \beta_3 - \beta_4$, $d = \beta_4 - \beta_1 + \beta$. A direct step-by-step integration gave

$$I_{\rho^2 \alpha^4} = \beta^3 \left(\frac{1}{48} - \frac{\tilde{\zeta}(3)}{16} \right) \quad (4)$$

that is, a linear combination of 1 and $\tilde{\zeta}(3) = 7\zeta(3)/2$ with rational coefficients. One inferred by connecting the integral (4) to the partition function that

$$\frac{\langle Z_2 \rangle}{Z_0} - 2 \frac{\langle Z_1 \rangle}{Z_0} + 1 = \frac{1}{Z_0^2} \left[\frac{1}{6} \alpha^2 + \left(\frac{1}{24} - \frac{\tilde{\zeta}(3)}{8} \right) \alpha^4 + \dots \right] \quad (5)$$

Thus, an Euler sum of level 3 emerges, which fits into the general scheme of Feynman diagram expansion in perturbative field theory [3], where Euler sums are known to play a central role. These sums are defined as

$$\zeta(n_1, n_2, \dots, n_p) = \sum_{k_i > k_{i+1} \geq 1} \prod_{i=1}^p \frac{(\pm)^{k_i}}{k_i^{n_i}}$$

where $n = n_1 + n_2 + \dots + n_p$ is the level of the sum. At level n , the simplest sums are $\zeta(n) = \sum_{k=1}^\infty 1/k^n$ and $\zeta_a(n) = \sum_{k=1}^\infty (-1)^k/k^n$, with $(2^n - 1)\zeta(n)/2^n = (\zeta(n) - \zeta_a(n))/2$.

On the other hand, Eqs. (3)–(4) imply that the nontrivial part of $I_{\rho^2 \alpha^4}$ is

$$\int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \int_0^{\beta_2} d\beta_3 \int_0^{\beta_3} d\beta_4 \frac{(a+c)(b+d)}{abc + bcd + cda + dab} = \beta^3 \frac{1 + \tilde{\zeta}(3)}{16}$$

In Ref. [4], algebraic manipulations and a Laplace transform with respect to β let one factorize this multiple integral as

$$\frac{1}{2} \int_{a,b,c,d=0}^\infty da db dc dd \int_0^\infty dt \frac{1}{cd} e^{-(a+b+c+d)-t(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})} = \int_0^\infty u K_0(u)^2 (u K_1(u))^2 du$$

where $K_\nu(u)$ are the modified Bessel functions ($t = u^2/2$). Consequently,

$$\int_0^\infty u K_0(u)^2 (u K_1(u))^2 du = \frac{1 + \tilde{\zeta}(3)}{16} \quad (6)$$

More generally, it was shown in [4] by means of a simple integration by parts that any integral of the form

$$\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du \quad (7)$$

with weight⁴ $p + q = 4$ and n even, is a linear combination with rational coefficients of $\tilde{\zeta}(3)$ and 1. This was achieved via a 2×2 matrix recurrence relation for the integrals $\int_0^\infty u^{n+1} K_0(u)^4 du$ and $\int_0^\infty u^{n+1} K_1(u)^4 du$, with $n \geq 4$ (see Sec. 3 for a derivation of this recurrence and its straightforward generalization to any weight $k = p + q$). A remarkable result is the fact that the initial condition

$$\int_0^\infty u^5 K_0(u)^4 du = \frac{-27 + 7\tilde{\zeta}(3)}{64}, \quad \int_0^\infty u^5 K_1(u)^4 du = \frac{53 - 9\tilde{\zeta}(3)}{64} \quad (8)$$

happens to be such that the asymptotic behavior of the recurrence relation for large n is governed by the smaller of the two eigenvalues $\{1/16, 1/4\}$ of the asymptotic recurrence matrix.

In Ref. [5], these considerations were extended to integrals involving K_ν as well as I_ν — specifically, the set (7) where either one of the K_0 's is replaced by I_0 or one of the K_1 's is replaced by I_1 (so one has a product of three K_ν 's and one I_ν). These integrals were shown, again via an elementary integration by parts, to be linear combinations with rational coefficients of $3\zeta(2)$ and 1. The 2×2 recurrence matrix is identical to the one obtained in the previous case, up to a minus sign in the off-diagonal elements. The initial condition

$$\int_0^\infty u^5 K_0(u)^3 I_0(u) du = \frac{21\zeta(2)}{128}, \quad \int_0^\infty u^5 K_1(u)^3 I_1(u) du = \frac{27\zeta(2)}{128} \quad (9)$$

is such that the asymptotic behavior is governed, as expected, by the bigger of the eigenvalues $\{1/16, 1/4\}$ — it does not for sure coincide with the unique initial condition (8) associated with the smallest eigenvalue.

2 The $\rho^2\alpha^6$ diagram with maximal impurity line crossing

Integrals of products of modified Bessel functions appear to play a central role in the perturbative analysis of the 2-impurity problem. Indeed, whereas the $\rho^2\alpha^5$ diagrams can be easily shown to reduce to $\rho^2\alpha^4$ diagrams, the relevant $\rho^2\alpha^6$ diagram with maximal impurity line crossing is much more arduous to compute. Following the same route as for the $\rho^2\alpha^4$ diagram, and again taking a Laplace transform, one has obtained [6] the expression

$$\begin{aligned} I_{\rho^2\alpha^6} = & 8 \int_0^\infty du u K_0(u)^2 (u K_1(u))^2 \int_0^u dx (x K_1(x)) I_1(x) K_0(x)^2 \\ & - 4 \int_0^\infty du u K_0(u) (u K_1(u)) [(u K_1(u)) I_0(u) - u K_0(u) I_1(u)] \int_u^\infty dx x K_0(x)^2 K_1(x)^2 \\ & + \int_0^\infty u K_0(u)^4 (u K_1(u))^2 du \end{aligned} \quad (10)$$

⁴Here and in the sequel we define the weight of an integral of a product of Bessel functions K_ν as the total power of the K_ν 's.

It contains not only, in analogy with order $\rho^2\alpha^4$, the integral

$$\int_0^\infty u K_0(u)^4 (u K_1(u))^2 du$$

of weight 6, but also a particular combination of nested double integrals of products of modified Bessel functions K_ν and I_ν . Components of those double integrals, if integrated individually from 0 to ∞ , are, as already demonstrated⁵, linear combinations of either $\zeta(3)$ and 1 or of $3\zeta(2)$ and 1. For example, in addition to Eq. (6), one has

$$\begin{aligned} \int_0^\infty (u K_1(u)) I_1(u) K_0(u)^2 du &= \frac{\zeta(2)}{8} \\ \int_0^\infty u K_0(u) (u K_1(u))^2 I_0(u) du &= \frac{8 + 3\zeta(2)}{32} \\ \int_0^\infty u^2 K_0(u)^2 (u K_1(u)) I_1(u) du &= \frac{8 - 3\zeta(2)}{32} \end{aligned}$$

In effect, there is a mapping, via an integral, of a product of K_ν —or a product of K_ν and I_ν —onto a linear combination with rational coefficients of $\zeta(3)$ and 1—or $\zeta(2)$ and 1, respectively:

$$(f) \rightarrow \int_0^\infty f(u) du = \zeta[f]$$

For the double integrals, the same scheme is at work, but now the mapping is to a “polyzeta” object:

$$(f, g) \rightarrow \int_0^\infty f(u) du \int_0^u g(x) dx = \zeta[f, g]$$

Since

$$\int_0^\infty f(u) du \int_0^u g(x) dx = \int_0^\infty f(u) du \int_0^\infty g(x) dx - \int_0^\infty g(u) du \int_0^u f(x) dx$$

one has

$$\zeta[f, g] = \zeta[f]\zeta[g] - \zeta[g, f]$$

in analogy with the relation involving the standard polyzeta function $\zeta(p, q) = \sum_{n>m} \frac{1}{n^p} \frac{1}{m^q}$:

$$\zeta(p, q) = \zeta(p)\zeta(q) - \zeta(p+q) - \zeta(q, p) \quad (11)$$

By analogy with lower orders, one might expect the $\rho^2\alpha^6$ corrections to be a linear combination with rational coefficients of Euler sums up to a certain level. The structure of the double integral in Eq. (10) clearly indicates that the highest level should be 5. Indeed, the constituent single integrals in the first and second terms reduce to levels 3 and 2, respectively; by virtue of Eq. (11), the product $\zeta(3)\zeta(2)$ is associated with $\zeta(5)$. The last term in (10), $\int_0^\infty u K_0(u)^4 (u K_1(u))^2 du$, is a Bessel integral of weight 6. Given that an integral (6) of weight 4 is related to $\zeta(3)$, i.e., level 3, this agains hints at level 5 in the case at hand.

However, a search for an integer relation with the PSLQ algorithm [7] does not confirm this expectation. On the contrary, it indicates that

$$I_{\rho^2\alpha^6} = \frac{1}{30} \int_0^\infty u K_0(u)^6 du + \frac{1}{20} \int_0^\infty u^3 K_0(u)^6 du - \frac{31}{160} \zeta(5) \quad (12)$$

⁵Note, however, that $\int_u^\infty x K_0(x)^2 K_1(x)^2 dx$ would be divergent if one set $u = 0$.

is a linear combination with rational coefficients of not only, as expected, a level 5 Euler sum $(2^5 - 1)\zeta(5)$, but also of two numbers of weight 6, $\int_0^\infty u K_0(u)^6 du$ and $\int_0^\infty u^3 K_0(u)^6 du$, neither of which is a linear combination with rational coefficients of Euler sums of level 5.

Moreover, the same PSLQ search shows that the last term in Eq. (10) is a linear combination of the same two numbers:

$$\int_0^\infty u K_0(u)^4 (u K_1(u))^2 du = \frac{2}{15} \int_0^\infty u K_0(u)^6 du - \frac{1}{5} \int_0^\infty u^3 K_0(u)^6 du \quad (13)$$

Hence, also the sum of double integrals in (10) is by itself a linear combination with rational coefficients of the same three numbers that appear in (12)—specifically, $-\int_0^\infty u K_0(u)^6 du/10 + \int_0^\infty u^3 K_0(u)^6 du/4 - 31\zeta(5)/160$.

3 Integrals $\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du$ with even n

Equation (13) for weight 6, together with the integrals of weight 4 in [4, 5], are but particular cases of a much more general pattern involving integrals of the form $\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du$ with even n .

At a given weight $k = p + q$, denote

$$I_{n,j}^{(k)} = \frac{1}{n!} \int_0^\infty u^{n+1} K_0(u)^{k-j} K_1(u)^j du \quad (14)$$

where $j = 0, 1, 2, \dots, k$. For the integral to be finite, $n \geq j - 1$ is required. Integration by parts, taking into account that $dK_0(u)/du = -K_1(u)$ and $d(uK_1(u))/du = -uK_0(u)$, gives a recurrence relation

$$I_{n,j}^{(k)} = \frac{n+1}{n-j+2} \left[j I_{n+1,j-1}^{(k)} + (k-j) I_{n+1,j+1}^{(k)} \right] \quad (15)$$

The mapping $\{I_{n+1,j}^{(k)}\} \rightarrow \{I_{n,j}^{(k)}\}$ induced by Eq. (15) and valid⁶ for $n \geq k$ involves a tridiagonal matrix

$$A_n^{(k)} = \begin{pmatrix} 0 & \frac{k(n+1)}{n+2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & k-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{2(n+1)}{n} & 0 & \frac{(k-2)(n+1)}{n} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{3(n+1)}{n-1} & 0 & \frac{(k-3)(n+1)}{n-1} & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{4(n+1)}{n-2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{k(n+1)}{n-k+2} & 0 \end{pmatrix} \quad (16)$$

One has $\det A_n^{(k)} = -\frac{(3)^2(5)^2 \dots (k)^2(n+1)^{2k+1}}{(n+2)(n+1)(n) \dots (n-k+2)}$ when k is odd, and 0 when k is even, meaning that in the latter case the $I_{n,j}^{(k)}$'s are linearly related. Indeed, for an even k , one has

$$\sum_{l=0}^{k/2} (-1)^l (n-2l+2) \binom{k/2}{l} I_{n,2l}^{(k)} = 0 \quad (17)$$

⁶We choose here, as a matter of convenience, to start the recurrence at $n = k$.

By applying relation (15) twice, one obtains a mapping $\{I_{n+2,j}^{(k)}\} \rightarrow \{I_{n,j}^{(k)}\}$:

$$\begin{aligned}
I_{n,j}^{(k)} = & \frac{(n+1)(n+2)(j-1)j}{(n-j+2)(n-j+4)} I_{n+2,j-2}^{(k)} \\
& + \frac{(n+1)(n+2)}{n-j+2} \left[\frac{(j+1)(k-j)}{n-j+2} + \frac{j(k-j+1)}{n-j+4} \right] I_{n+2,j}^{(k)} \\
& + \frac{(n+1)(n+2)(k-j-1)(k-j)}{(n-j+2)^2} I_{n+2,j+2}^{(k)}
\end{aligned} \tag{18}$$

This can be inverted, as long as takes into account the linear relation (17) in the case of an even k .

The recurrence relations (15), (18) conserve the parity of $n-j$; thus, all $I_{n,j}^{(k)}$'s are divided into two subsets, and the relations operate separately within each subset. For reasons that will become clear later, we focus on one of those—the one for which $n-j$ is even. To span this subset, it is enough to assume that n is even, then take the set

$$I_{n,0}^{(k)}, \quad I_{n,2}^{(k)}, \quad \dots, \quad I_{n,k}^{(k)}, \quad n \geq k \quad (k \text{ even}) \tag{19}$$

[remembering that these integrals are linearly related via Eq. (17)], or

$$I_{n,0}^{(k)}, \quad I_{n,2}^{(k)}, \quad \dots, \quad I_{n,k-1}^{(k)}, \quad n \geq k-1 \quad (k \text{ odd}) \tag{20}$$

and note that the integrals

$$I_{n+1,1}^{(k)}, \quad I_{n+1,3}^{(k)}, \quad \dots, \quad I_{n+1,k-1}^{(k)} \quad (k \text{ even}) \tag{21}$$

and

$$I_{n+1,1}^{(k)}, \quad I_{n+1,3}^{(k)}, \quad \dots, \quad I_{n+1,k}^{(k)} \quad (k \text{ odd}) \tag{22}$$

are related to (19) and (20), respectively, by inverting the recurrence (15) [again, taking into account (17) if k even]. The union of sets (19)–(22), at a given k , is tantamount to the family of integrals

$$\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du \tag{23}$$

with weight $p+q=k$ and an even n .

For an even k , by virtue of Eq. (18), all integrals from the family (19), (21) can be expressed as a linear combination with rational coefficients of the initial conditions

$$I_{k,0}^{(k)}, \quad I_{k,2}^{(k)}, \quad \dots, \quad I_{k,k}^{(k)} \tag{24}$$

still taking into account Eq. (17). Noting further that Eq. (15) implies $2I_{k-1,k-1}^{(k)} = k(I_{k,k}^{(k)} + (k-1)I_{k,k-2}^{(k)})$ and that, trivially, $I_{k-1,k-1}^{(k)} = 1/k!$, one can trade $I_{k,k}^{(k)}$ for 1. Using (17), one can get rid of one more element: thus, all integrals (19), (21) are linear combinations with rational coefficients of a basis made of the $k/2$ independent (in the sense that none of them is a linear combination with rational coefficients of the others) numbers

$$\{1, I_{k,0}^{(k)}, I_{k,2}^{(k)}, \dots, I_{k,k-4}^{(k)}\} \tag{25}$$

Likewise, for k odd, starting with the initial conditions

$$I_{k-1,0}^{(k)}, \quad I_{k-1,2}^{(k)}, \quad \dots, \quad I_{k-1,k-1}^{(k)} \quad (26)$$

and again using $I_{k-1,k-1}^{(k)} = 1/k!$, one concludes that any integral in (20), (22) is a linear combination with rational coefficients of the basis made of the $(k+1)/2$ independent numbers

$$\{1, I_{k-1,0}^{(k)}, I_{k-1,2}^{(k)}, \dots, I_{k-1,k-1}^{(k)}\} \quad (27)$$

Last but not least, one can also show, by applying (15) appropriately for $0 \leq n \leq k$, that for k even, the basis (25) can be mapped on the basis

$$\{1, I_{0,0}^{(k)}, I_{2,0}^{(k)}, \dots, I_{k-4,0}^{(k)}\} \quad (28)$$

and for k odd, the basis (27) can mapped on

$$\{1, I_{0,0}^{(k)}, I_{2,0}^{(k)}, \dots, I_{k-3,0}^{(k)}\} \quad (29)$$

Therefore, any integral in the set (23) is a linear combination with rational coefficients of the basis (28) or (29), for k even or odd, respectively.

Consider now the asymptotic regime, $n \rightarrow \infty$. In that limit, Eq. (18) becomes

$$I_{n,j}^{(k)} = A_{j,j-2}^{(k)} I_{n+2,j-2}^{(k)} + A_{j,j}^{(k)} I_{n+2,j}^{(k)} + A_{j,j+2}^{(k)} I_{n+2,j+2}^{(k)} \quad (30)$$

where the elements of the tridiagonal transition matrix $A^{(k)}$ (indexed so that their subscripts are always even) no longer depend on n :

$$A_{j,j-2}^{(k)} = (j-1)j, \quad A_{j,j}^{(k)} = 2j(k-j) + k, \quad A_{j,j+2}^{(k)} = (k-j-1)(k-j) \quad (31)$$

The $[(k-k \bmod 2)/2 + 1]$ eigenvalues of this matrix are: $k^2, (k-2)^2, (k-4)^2, \dots$; the last one is 1 for odd k and 0 for even k . In the latter case, before inverting the matrix, one has to reduce its dimension by one, using Eq. (17). Thereupon, the largest eigenvalue of the inverse matrix is 1 or $1/4$ for k odd or even, respectively, whereas the smallest one is $1/k^2$. Remarkably, as we discovered experimentally, with the initial conditions (24), (26) it is the smallest eigenvalue that determines the asymptotic behavior of $I_{n,j}^{(k)}$. In the asymptotic regime itself, this can be understood by noting that the eigenvector corresponding to said eigenvalue, i.e., to the k^2 eigenvalue of $A^{(k)}$, is $\{1, 1, \dots, 1\}$ — because $A_{j,j-2}^{(k)} + A_{j,j}^{(k)} + A_{j,j+2}^{(k)} = k^2$. Now, in the limit $n \rightarrow \infty$, the integral $I_{n,j}^{(k)}$ does not depend on j , because the integrand $u^{n+1} K_0(u)^{k-j} K_1(u)^j$ peaks at large values of u , where both $K_0(u)$ and $K_1(u)$ can be approximated by their common asymptotic behavior, $K_\nu(u) \xrightarrow{u \rightarrow \infty} \sqrt{\frac{\pi}{2u}} e^{-u}$. Therefore, the vector $\{I_{n,j}^{(k)}\}$ becomes, in the asymptotic limit, proportional to $\{1, 1, \dots, 1\}$. The fact that the initial condition (24), (26) leads to this special asymptotic behavior, as well as the fact that said condition can be expressed in terms of the basis (28)–(29), means that the building blocks of (28)–(29), namely Bessel function integrals $\int_0^\infty u^{n+1} K_0(u)^k du$ with n even, may play some special role in number theory—like Euler sums do (but again, these integrals are not rational linear combinations of those sums).

As an example, we detail the recurrence relations for weight $k = 3$, where one has computed [4]

$$I_{0,0}^{(3)} = \frac{1}{6} \sum_{k=0}^{\infty} \frac{1}{(k+1/3)^2} - \frac{2}{3} \zeta(2) \quad (32)$$

One has, for an even $n \geq 2$,

$$\begin{pmatrix} I_{n,0}^{(3)} \\ I_{n,2}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{3(n+1)}{n+2} & \frac{6(n+1)}{n+2} \\ \frac{2(n+1)}{n} & \frac{(n+1)(7n+6)}{n^2} \end{pmatrix} \begin{pmatrix} I_{n+2,0}^{(3)} \\ I_{n+2,2}^{(3)} \end{pmatrix} \quad (33)$$

The inverse relation is

$$\begin{pmatrix} I_{n+2,0}^{(3)} \\ I_{n+2,2}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{7n+6}{9(n+1)} & -\frac{2n^2}{3(n+1)(n+2)} \\ -\frac{2n}{9(n+1)} & \frac{n^2}{3(n+1)(n+2)} \end{pmatrix} \begin{pmatrix} I_{n,0}^{(3)} \\ I_{n,2}^{(3)} \end{pmatrix} \quad (34)$$

which in the asymptotic limit turns into

$$\begin{pmatrix} I_{n+2,0}^{(3)} \\ I_{n+2,2}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{7}{9} & -\frac{2}{3} \\ -\frac{2}{9} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} I_{n,0}^{(3)} \\ I_{n,2}^{(3)} \end{pmatrix} \quad (35)$$

with eigenvalues $\{1/9, 1\}$.

Clearly, as alluded to above, instead of the set (23), one could have focused on the set $\int_0^\infty u^{n+1} K_0(u)^p (u K_1(u))^q du$ with weight $p+q=k$ and n odd—corresponding to the subset of integrals (14) with odd $n-j$. The recurrence still operates within this set, and the same kind of algebra as above is at work. However, the integrals in this set

- do not play any role in the perturbative analysis of the random magnetic impurity problem (at least up to 6th order);
- but still lead to an asymptotic behavior governed by the smallest eigenvalue of the corresponding asymptotic recurrence matrix. This can be easily seen in the case $k=3$, where the initial conditions $I_{2,3}^{(3)}$, $I_{2,1}^{(3)}$ for the integrals $I_{n,3}^{(3)}$ and $I_{n,1}^{(3)}$ with n even and the corresponding recurrence relation lead to an asymptotic governed by the smallest of the eigenvalues $\{1/9, 1\}$ of the asymptotic matrix.

Note finally that at weight $k=n+1$, by singling out the simplest integral $I_{0,0}^{(n+1)}$ in the basis (28) or (29), one arrives at the number

$$\kappa(n) = \frac{1}{(n+1)!} \int_0^\infty u K_0(u)^{n+1} du \quad (36)$$

with $\kappa(0) = 1$; $\kappa(1) = 1/4$; $\kappa(2) = I_{0,0}^{(3)}/3!$, see Eq. (32); $\kappa(3) = 7\zeta(3)/(8 \times 4!)$, etc. As already said, this number is analogous to Euler sums of level n , but it is not a rational linear combination of those. When $n \rightarrow \infty$, the $1/(n+1)!$ normalization in (36) is such that⁷

$$e^{-2\Gamma'(1)} \lim_{n \rightarrow \infty} \kappa(n) = \lim_{n \rightarrow \infty} (\zeta(n) - 1) \quad (37)$$

When $n \rightarrow -1$, on the other hand, $\kappa(n) = 1/(1+n)^2$ plus logarithmic subleading terms and a constant. Clearly, the function $\kappa(n)$ defined for $n \geq -1$ real can be analytically continued to the function $\kappa(s)$ defined on the whole complex half-plane $\Re(s) \geq -1$.

⁷This can be easily shown by recognizing that when $n \rightarrow \infty$, the integrand $u K_0(u)^{n+1}$ has a peak near the origin, where $K_0(u)$ can be approximated by $K_0(u) \simeq -\log \frac{u}{2 \exp[\Gamma'(1)]}$. It follows that the main contribution to the integral is $\int_0^\infty u K_0(u)^{n+1} du \simeq \int_0^{2 \exp[\Gamma'(1)]} u (-\log \frac{u}{2 \exp[\Gamma'(1)]})^{n+1} du$, which trivially yields (37).

4 Conclusion

We have demonstrated that a class of integrals involving Bessel functions, which arise in perturbation theory—in particular, in the two-dimensional problem of random magnetic impurities—can be expressed, via recurrence relations, as linear combinations with rational coefficients of “basis” integrals. Some of the latter, in turn, reduce to Euler sums, but most do not. Additionally, these same basis integrals, for an even power of the argument, turn out to generate the unique initial conditions for the recurrence relations which make the smallest eigenvalue of the transition matrix determine the asymptotic behavior. The nature of this phenomenon has yet to be understood more deeply.

It is not only single but also some double nested integrals that happen to be linear combinations with rational coefficients of said basis integrals: one that does is the integral that figures in the 6th order of perturbation theory in the physical problem at hand. Understanding what other double integrals fall into the same class remains another open question.

Acknowledgements: One of us (S.O.) would like to thank Jean Desbois for discussions and some technical help, in particular at the end of Section 3.

Note added: After completion of this work, we became aware of Ref. [8] and in particular of Ref. 26 therein, where some results overlap with those of section 3.

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